

Full and maximal squashed flat antichains of minimum weight

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Abstract

A full squashed flat antichain (FSFA) in the Boolean lattice B_n is a family $\mathcal{A} \cup \mathcal{B}$ of subsets of $[n] = \{1, 2, \dots, n\}$ such that, for some $k \in [n]$ and $0 \leq m \leq \binom{n}{k}$, \mathcal{A} is the family of the first m k -sets in squashed (reverse-lexicographic) order and \mathcal{B} contains exactly those $(k-1)$ -subsets of $[n]$ that are not contained in some $A \in \mathcal{A}$. If, in addition, every k -subset of $[n]$ which is not in \mathcal{A} contains some $B \in \mathcal{B}$, then $\mathcal{A} \cup \mathcal{B}$ is a maximal squashed flat antichain (MSFA). For any n, k and positive real numbers α, β , we determine all FSFA and all MSFA of minimum weight $\alpha \cdot |\mathcal{A}| + \beta \cdot |\mathcal{B}|$. Based on this, asymptotic results on MSFA with minimum size and minimum BLYM value, respectively, are derived.

Keywords: Antichain, Sperner family, Flat Antichain Theorem, Kruskal-Katona Theorem, BLYM inequality

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1 Introduction

An *antichain* in the Boolean lattice B_n is a family of subsets of $[n] := \{1, 2, \dots, n\}$ such that none of the subsets is properly contained in another. An antichain $\mathcal{F} \subseteq B_n$ is *flat* if $|F| \in \{k-1, k\}$ for all $F \in \mathcal{F}$ and some $1 \leq k \leq n$. In this paper, we study flat antichains with the property that no $(k-1)$ -set can be added without destroying the antichain property and such that the k -sets form an initial segment in squashed (or colexicographic) order. Such *full squashed flat antichains* (FSFA) are known to generate ideals of minimum size among all antichains of the same size in B_n (Clements [2]). This fact and the Flat Antichain Theorem are the main motivations for this research. More detailed explanations are given later in this introductory section and in the beginning of the next section. Our main result is a characterization of those FSFA that attain minimum weight with respect to certain weight functions. In particular, we determine all FSFA of minimum size, minimum volume, and minimum BLYM value, respectively.

Throughout, let n be a positive integer. We use $2^{[n]}$ or B_n to denote the power set of $[n]$ and $\binom{[n]}{i}$ for the family of all i -subsets of $[n]$. The *volume* $V(\mathcal{F})$ of $\mathcal{F} \subseteq 2^{[n]}$ is defined as $V(\mathcal{F}) := \sum_{F \in \mathcal{F}} |F|$. The results of Kisvölcsy [9] and Lieby [11, 12] perfectly complement each other to give the following theorem.

Theorem 1 (Flat Antichain Theorem (FLAT)). *Let $\mathcal{F} \subseteq 2^{[n]}$ be an antichain. Then there is a flat antichain $\mathcal{F}' \subseteq 2^{[n]}$ with $|\mathcal{F}'| = |\mathcal{F}|$ and $V(\mathcal{F}') = V(\mathcal{F})$.*

FLAT can be nicely stated using the following equivalence relation on the set of all antichains in B_n . We say that two antichains are equivalent if they have the same size and the same volume. By Theorem 1, each of the equivalence classes with respect to this relation contains some flat antichain. Proposition 2 below illustrates that flat antichains are in some sense the extremal representatives of their equivalence classes. Let \mathbb{R}^+ denote the set of nonnegative real numbers, and consider a weight function $w : 2^{[n]} \mapsto \mathbb{R}^+$ such that each i -set $F \subseteq [n]$ has the same weight $w(F) = w_i$. The weight of $\mathcal{F} \subseteq 2^{[n]}$ is defined to be $w(\mathcal{F}) = \sum_{F \in \mathcal{F}} w(F)$. The sequence $\{w_i\}_{i=0}^n$ is *convex* if $\{w_i - w_{i-1}\}_{i=1}^n$ is increasing and *concave* if $\{w_i - w_{i-1}\}_{i=1}^n$ is decreasing.

Proposition 2. *Let $w : 2^{[n]} \mapsto \mathbb{R}^+$ be a weight function as above. Furthermore, let $\mathcal{A} \subseteq 2^{[n]}$ be an antichain and $\mathcal{F} \subseteq 2^{[n]}$ a flat antichain such that $|\mathcal{F}| = |\mathcal{A}|$ and $V(\mathcal{F}) = V(\mathcal{A})$.*

- (i) *If the sequence $\{w_i\}_{i=0}^n$ is convex, then $w(\mathcal{F}) \leq w(\mathcal{A})$.*
- (ii) *If the sequence $\{w_i\}_{i=0}^n$ is concave, then $w(\mathcal{F}) \geq w(\mathcal{A})$.*

Proof: We only prove part (i) here. The proof of (ii) is analogous.

Assume that $\{w_i\}_{i=0}^n$ is convex. Let $\mathbf{a} = (a_0, a_1, \dots, a_n)$ be the profile vector of \mathcal{A} , i.e., $a_i = |\{A \in \mathcal{A} : |A| = i\}|$. Furthermore, let $\ell = \min\{i : a_i \neq 0\}$

and $u = \max\{i : a_i \neq 0\}$. The weight of \mathcal{A} is determined by \mathbf{a} , and one has $w(\mathcal{A}) = \sum_{i=\ell}^u a_i w_i$ for which we also write $w(\mathbf{a})$. If $u - \ell \leq 1$, then \mathcal{A} is flat. As \mathcal{A} and \mathcal{F} have the same size and the same volume, their profile vectors must then be equal, and it follows that $w(\mathcal{A}) = w(\mathcal{F})$. Hence, without loss of generality we can assume that $u - \ell \geq 2$.

Consider the vector \mathbf{a}' obtained from \mathbf{a} replacing a_ℓ by $a_\ell - 1$, $a_{\ell+1}$ by $a_{\ell+1} + 1$, a_u by $a_u - 1$, and a_{u-1} by $a_{u-1} + 1$. (That is, if $u - \ell = 2$, then $a_{\ell+1} = a_{u-1}$ will be increased by 2.) Note that $\sum a'_i = \sum a_i$, $\sum a'_i i = \sum a_i i$, and

$$w(\mathbf{a}) - w(\mathbf{a}') = (w_u - w_{u-1}) - (w_{\ell+1} - w_\ell).$$

As $\{w_i\}$ is convex, it follows that $w(\mathbf{a}') \leq w(\mathbf{a})$. (It should be pointed out that we do not claim or need that \mathbf{a}' is the profile vector of some antichain in B_n .) Iterating this process, we transform \mathbf{a} into the profile vector \mathbf{f} of \mathcal{F} as \mathcal{A} and \mathcal{F} agree in size and volume. This implies $w(\mathcal{F}) = w(\mathbf{f}) \leq w(\mathbf{a}) = w(\mathcal{A})$. ■

The well-known BLYM inequality (see [3] for instance) states that the *BLYM value* (also known as the *Lubell function*, cf. [6]) of any antichain in B_n is at most 1, where the BLYM value of $\mathcal{F} \subseteq 2^{[n]}$ is defined to be $\sum_{F \in \mathcal{F}} 1/\binom{n}{|F|}$. In this context, Proposition 2 implies an interesting observation about flat antichains.

Corollary 3. *Flat antichains have minimum BLYM values within their equivalence classes.*

Proof: The claim follows from Proposition 2 and the fact that the sequence $\{1/\binom{n}{i}\}_{i=0}^n$ is convex which is straightforward to verify. ■

For a family $\mathcal{G} \subseteq \binom{[n]}{i}$ the *shadow* and the *shade* (or *upper shadow*) of \mathcal{G} are the families $\Delta\mathcal{G} := \{H \in \binom{[n]}{i-1} : H \subset G \text{ for some } G \in \mathcal{G}\}$ and $\nabla\mathcal{G} := \{H \in \binom{[n]}{i+1} : H \supset G \text{ for some } G \in \mathcal{G}\}$, respectively. $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$ with $\mathcal{A} \subseteq \binom{[n]}{k}$ and $\mathcal{B} \subseteq \binom{[n]}{k-1}$ for some $1 \leq k \leq n$ is called a *full flat antichain (FFA)* if $\mathcal{B} = \binom{[n]}{k-1} \setminus \Delta\mathcal{A}$. *Maximal flat antichains (MFA)* are the ones that in addition satisfy $\mathcal{A} = \binom{[n]}{k} \setminus \nabla\mathcal{B}$.

For example, $\mathcal{F} = \{\{1, 2\}, \{1, 3\}, \{4\}\}$ is an FFA in B_4 as all singletons other than $\{4\}$ are covered by the 2-sets in \mathcal{F} . On the other hand, \mathcal{F} is not an MFA since $\{2, 3\}$ could be added in without destroying the antichain property.

It is easy to see that \mathcal{F} is an MFA if and only if its complementary antichain $\overline{\mathcal{F}} := \{[n] \setminus F : F \in \mathcal{F}\}$ is an MFA. If \mathcal{F} is an FFA, then $\overline{\mathcal{F}}$ is an FFA only if \mathcal{F} is an MFA.

In [5], it is shown that the minimum size of an MFA with $k = 3$ (that is an MFA consisting of 2-sets and 3-sets) is $\binom{n}{2} - \lfloor (n+1)^2/8 \rfloor$, and all such MFA of minimum size are determined.

Actually, in [5] the more general problem of minimising the total weight of an MFA with $k = 3$ was solved when all 3-sets have weight α and all 2-sets have weight β for some positive constants α and β . Some further results are obtained in [7] for the corresponding problem in the more general setting where the antichain is restricted to contain only sets with cardinalities in some given set K , where the flat case corresponds to $K = \{k, k - 1\}$.

In the next section, we solve the problem similar to the one in [5] for squashed FFA and squashed MFA for any n and k .

2 FSFA and MSFA of minimum weight

Following Anderson [1], we say that $F \subseteq [n]$ precedes $G \subseteq [n]$, $G \neq F$, in *squashed* (or *colexicographic*) order and write $F <_S G$ whenever $\max((F \cup G) \setminus (F \cap G)) \in G$.

A *full squashed flat antichain* (FSFA) in B_n is an FFA of the form $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$, where $\mathcal{A} \subseteq \binom{[n]}{k}$ and $\mathcal{B} \subseteq \binom{[n]}{k-1}$ for some $1 \leq k \leq n$ and such that \mathcal{A} consists of the first m elements of $\binom{[n]}{k}$ with respect to squashed order for some $m \leq \binom{n}{k}$. Clearly, an FSFA \mathcal{F} is completely determined by n , k , and m . If an FSFA \mathcal{F} is an MFA, then we call it a *maximal squashed flat antichain* (MSFA). Every FSFA $\mathcal{A} \cup \mathcal{B}$ is contained in a unique MSFA $\mathcal{A}' \cup \mathcal{B}$, see Proposition 6 below.

By Sperner's Theorem [13], any antichain in B_n has size at most $\binom{n}{\lfloor n/2 \rfloor}$. It is a remarkable fact that for any positive $s \leq \binom{n}{\lfloor n/2 \rfloor}$ there is an FSFA of size s in B_n . Moreover, for any s , among all antichains of size s in B_n there is a unique FSFA that generates an ideal of minimum weight, where each i -set in B_n has the same weight w_i and $0 \leq w_0 \leq w_1 \leq \dots \leq w_n$. For details, see Theorem 8.3.5 in Engel's book [3]. The last statement was generalized to Macaulay posets P with the property that P and its dual are weakly shadow increasing in [4].

For $1 \leq k \leq n$ and $0 \leq m \leq \binom{n}{k}$, the k -cascade representation of m is a representation of m in the form

$$m = \sum_{i=1}^k \binom{a_i}{i} \quad \text{with} \quad a_k > a_{k-1} > \dots > a_t \geq t > 0 = a_{t-1} = \dots = a_1. \quad (1)$$

The terms $\binom{a_i}{i}$ with $a_i = 0$ could clearly be removed from the above representation of m . Their only purpose here is that they will allow us a more compact formulation of the main result (Theorem 7). It is easy to see (cf. [8]) that for given k and m there is a unique k -cascade representation of m . Moreover, if \mathcal{A} is the family of the first m k -sets in squashed order and (1) is the k -cascade representation of m , then

$$|\Delta \mathcal{A}| = \sum_{i=t}^k \binom{a_i}{i-1}. \quad (2)$$

By the Kruskal-Katona Theorem [8, 10], the family \mathcal{A} of the first m k -sets in squashed order has a shadow of smallest size among all m -element subsets of $\binom{[n]}{k}$.

Although the following is well-known, we will provide a proof for completeness.

Proposition 4. *Let $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$ be an FSFA with $\mathcal{A} \subseteq \binom{[n]}{k}$ and $\mathcal{B} \subseteq \binom{[n]}{k-1}$, where $k \geq 2$, and let $m := |\mathcal{A}|$ be represented as in (1). \mathcal{F} is an MSFA if and only if $a_1 = 0$.*

Our proof of Proposition 4 makes use of the following lemma.

Lemma 5. *Let \mathcal{F} , \mathcal{A} , \mathcal{B} , and m be as in Proposition 4. Furthermore, assume that $\mathcal{A} = \{A_1, \dots, A_m\}$, where the sets are listed in squashed order, and that $A_m = \{x_1, x_2, \dots, x_k\}$ with $x_1 < x_2 < \dots < x_k$. \mathcal{F} is an MSFA if and only if $x_2 = x_1 + 1$.*

Proof: Assume that $\mathcal{B} = \{B_1, \dots, B_\ell\}$, where the sets are listed in squashed order. Then B_1 is the successor of $\{x_2, \dots, x_k\}$ in $\binom{[n]}{k-1}$ with respect to squashed order.

Let i be the largest index with $x_i = x_1 + i - 1$, i.e., $x_2 = x_1 + 1$ if and only if $i \geq 2$. If $x_i = n$, then $\mathcal{A} = \binom{[n]}{k}$ and we are done. So assume that $x_i < n$.

If $i \geq 2$, then $B_1 = \{1, 2, \dots, i-2, x_i+1, x_{i+1}, x_{i+2}, \dots, x_k\}$, and $\nabla \mathcal{B}$ contains

$$\{1, 2, \dots, i-2, i-1, x_i+1, x_{i+1}, x_{i+2}, \dots, x_k\}$$

which precedes A_m in squashed order.

If $i = 1$, then the successor of A_m in squashed order on $\binom{[n]}{k}$ is $\{x_1+1, x_2, \dots, x_k\}$, which is not contained in $\nabla \mathcal{B}$, because B_1 comes after $\{x_2, \dots, x_k\}$, and hence, every element of $\nabla \mathcal{B}$ comes after $\{x_2-1, x_2, x_3, \dots, x_k\}$. ■

Proof (Proposition 4): The k -cascade representation of m yields that

$$\begin{aligned} \mathcal{A} = & \binom{[a_k]}{k} \cup \left\{ A \cup \{a_k + 1\} : A \in \binom{[a_{k-1}]}{k-1} \right\} \\ & \cup \left\{ A \cup \{a_{k-1} + 1, a_k + 1\} : A \in \binom{[a_{k-2}]}{k-2} \right\} \\ & \cup \dots \cup \left\{ A \cup \{a_2 + 1, a_3 + 1, \dots, a_k + 1\} : A \in \binom{[a_1]}{1} \right\}. \end{aligned}$$

Let i be the smallest index with $a_i > 0$. The last element of \mathcal{A} with respect to squashed order is $A_m = \{a_i - i + 1, \dots, a_i - 1, a_i, a_{i+1} + 1, \dots, a_k + 1\}$. If $i = 1$, then A_m starts with $a_1, a_2 + 1, \dots$, and if $i > 1$, then A_m starts with $a_i - i + 1, a_i - i + 2$. Now the claim follows by Lemma 5. ■

Proposition 6. *Let $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$ be an FSFA with $\mathcal{A} \subseteq \binom{[n]}{k}$ and $\mathcal{B} \subseteq \binom{[n]}{k-1}$, where $k \geq 2$. \mathcal{F} is contained in a unique MSFA $\mathcal{F}' \subseteq \binom{[n]}{k} \cup \binom{[n]}{k-1}$, and \mathcal{F}' is of the form $\mathcal{A}' \cup \mathcal{B}$ with $\mathcal{A} \subseteq \mathcal{A}' \subseteq \binom{[n]}{k}$.*

Proof: Let \mathcal{A}' be the largest initial segment of $\binom{[n]}{k}$ with respect to squashed order such that $\Delta\mathcal{A}' = \Delta\mathcal{A}$. Clearly, $\mathcal{F}' = \mathcal{A}' \cup \mathcal{B}$ is an MSFA and $\mathcal{A} \subseteq \mathcal{A}'$.

By the choice of \mathcal{A}' , no FSFA $\mathcal{A}'' \cup \mathcal{B}$ with $\mathcal{A} \subsetneq \mathcal{A}'' \subsetneq \mathcal{A}'$ is an MSFA, and no FSFA on $\binom{[n]}{k} \cup \binom{[n]}{k-1}$ that contains more than $|\mathcal{A}'|$ k -sets can have \mathcal{B} as a subset. ■

Our main result is the following characterization of all FSFA of minimum weight. To avoid certain technicalities, the trivial cases $k = 1$ and $k = n$ are excluded.

Theorem 7. *Let $1 < k < n$ be integers, α, β positive real numbers and $\lambda := \beta/\alpha$. Furthermore, let $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$ with $\mathcal{A} \subseteq \binom{[n]}{k}$ and $\mathcal{B} \subseteq \binom{[n]}{k-1}$ be an FSFA, and let (1) be the k -cascade representation of $m := |\mathcal{A}|$. \mathcal{F} has minimum weight $w(\mathcal{F}) = \alpha \cdot |\mathcal{A}| + \beta \cdot |\mathcal{B}|$ among all FSFA in $\binom{[n]}{k} \cup \binom{[n]}{k-1}$ if and only if*

$$a_i = \begin{cases} n - k - 1 + i & \text{if } i > 1 + (n - k)/\lambda, \\ \lceil (i - 1)(\lambda + 1) - 1 \rceil \text{ or } \lfloor (i - 1)(\lambda + 1) \rfloor & \text{if } 1 + (n - k)/\lambda \geq i \geq 1 + 2/\lambda, \\ i & \text{if } 1 + 2/\lambda > i > 1/\lambda, \\ 0 \text{ or } i & \text{if } 1/\lambda = i, \\ 0 & \text{if } 1/\lambda > i. \end{cases}$$

Proof: First, observe that with $g(m) := m - \lambda|\Delta\mathcal{A}|$ we have

$$w(\mathcal{F}) = \alpha \cdot g(m) + \beta \binom{n}{k-1}.$$

Hence, our problem of minimizing $w(\mathcal{F})$ is equivalent to minimizing $g(m)$ over all $m \in \{0, 1, \dots, \binom{n}{k}\}$.

If $m \in \{\binom{n}{k} - 1, \binom{n}{k}\}$, then $\Delta\mathcal{A} = \binom{[n]}{k-1}$ holds. Consequently, $m = \binom{n}{k}$ does not minimize $g(m)$, and we can assume that $m < \binom{n}{k}$, i.e. that $a_k \leq n - 1$. As $a_k > a_{k-1} > \dots > a_t$, this implies

$$a_i = 0 \text{ or } i \leq a_i \leq n - 1 - k + i \quad \text{for } i \in [k]. \quad (3)$$

By (2), we have

$$g(m) = \sum_{i=t}^k h_i(a_i), \quad (4)$$

where, for $i \in [k]$, the polynomial $h_i : \mathbb{R} \mapsto \mathbb{R}$ is defined by

$$h_i(x) := \binom{x}{i} - \lambda \binom{x}{i-1} = \begin{cases} x - \lambda & \text{if } i = 1, \\ \frac{x + 1 - i(\lambda + 1)}{i!} \prod_{j=0}^{i-2} (x - j) & \text{if } i \geq 2. \end{cases}$$

Our strategy is as follows: For each $i \in [k]$, we determine those $x \in [i, n - 1 - k + i] \cap \mathbb{Z}$ for which $h_i(x)$ is smallest possible. For such x , we choose $a_i = x$ or $a_i = 0$

if $h_i(x)$ is negative or positive, respectively. If $h_i(x) = 0$, we choose $a_i \in \{0, x\}$. Eventually, we will verify that, with the a_i 's chosen as described, we obtain a proper k -cascade representation (1), i.e., that the following implication is true:

$$(i \in [k-1]) \wedge (a_i > 0) \implies (a_i < a_{i+1}). \quad (5)$$

To begin with, note that $h_1(x) = x - \lambda$ attains its global minimum with respect to the interval $[1, n-k]$ at $x = 1$, and we have $h_1(1) = 1 - \lambda$.

Let $i \in [k] \setminus \{1\}$. Then h_i is a polynomial of degree i with leading coefficient 1 and zeros $0, 1, \dots, i-2$ and $i(\lambda+1) - 1$. That means, $h_i(x)$ is positive and strictly increasing for $x > i(\lambda+1) - 1$, and $h_i(x) < 0$ for $i-2 < x < i(\lambda+1) - 1$. Moreover, h_i is strictly convex on $I := (i-2, i(\lambda+1) - 1)$. The numbers $u := (i-1)(\lambda+1)$ and $u-1$ both lie in I , and one can easily check that $h_i(u-1) = h_i(u)$.

Based on the discussion above, we distinguish three cases to find the global minimum of $h_i(x)$ over all $x \in [i, n-1-k+i] \cap \mathbb{Z}$.

Case 1: Assume that $u-1 < i$. Note that this is equivalent to $i < 1 + 2/\lambda$. In this case, $h_i(x)$ is a minimum only at $x = i$, and $h_i(i)$ is positive if $i < 1/\lambda$, equals 0 if $i = 1/\lambda$ and is negative if $i > 1/\lambda$.

Case 2: Assume that $i \leq u-1$ and that $u \leq n-1-k+i$. Note that this is equivalent to $1 + 2/\lambda \leq i \leq 1 + (n-k)/\lambda$. In this case, $h_i(x)$ attains its minimum exactly for $x \in \{\lceil u-1 \rceil, \lfloor u \rfloor\}$, and this minimum is negative.

Case 3: Assume that $n-1-k+i < u$. Note that this is equivalent to $1 + (n-k)/\lambda < i$. In this case, $h_i(x)$ is a minimum only at $x = n-1-k+i$ and $h_i(n-1-k+i) < 0$.

By the results of the case-by-case analysis above and (4), $g(m)$ becomes a minimum when the a_i 's are chosen as in the theorem, where the minimization is over all choices satisfying (3). Finally, a straightforward calculation shows that (5) holds for the a_i 's as in the theorem. \blacksquare

Note that, by Proposition 4, for $\lambda < 1$ the optimal FSFA in Theorem 7 are also MSFA. In general, Theorem 7 and its proof yield the following characterization of minimum weight MSFA.

Corollary 8. *Let $1 < k < n$ be integers, α, β positive real numbers and $\lambda := \beta/\alpha$. Furthermore, let $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$ with $\mathcal{A} \subseteq \binom{[n]}{k}$ and $\mathcal{B} \subseteq \binom{[n]}{k-1}$ be an MSFA, and let (1) be the k -cascade representation of $m := |\mathcal{A}|$. \mathcal{F} has minimum weight $w(\mathcal{F}) = \alpha \cdot |\mathcal{A}| + \beta \cdot |\mathcal{B}|$ among all MSFA in $\binom{[n]}{k} \cup \binom{[n]}{k-1}$ if and only if*

(a) $\lambda \leq n - k + 1$ and

$$a_i = \begin{cases} n - k - 1 + i & \text{if } i > 1 + (n - k)/\lambda, \\ \lceil (i - 1)(\lambda + 1) - 1 \rceil \text{ or } \lfloor (i - 1)(\lambda + 1) \rfloor & \text{if } 1 + (n - k)/\lambda \geq i \geq 1 + 2/\lambda, \\ i & \text{if } 1 + 2/\lambda > i > \max\{1/\lambda, 1\}, \\ 0 \text{ or } i & \text{if } 1/\lambda = i > 1, \\ 0 & \text{otherwise,} \end{cases}$$

or

(b) $\lambda \geq n - k + 1$, $a_i = 0$ for $i = 1, \dots, k - 1$ and $a_k = n$.

Proof: In the beginning of the proof of Theorem 7 we ruled out the case that $\mathcal{F} = \binom{[n]}{k}$ when looking for FSFA of minimum weight. For $\lambda < n - k + 1$, the MSFA $\binom{[n]}{k}$ cannot be an MSFA of minimum weight either. This follows from the simple observation that in this case, the MSFA

$$\left(\binom{[n]}{k} \setminus \nabla \{ \{n - k + 2, n - k + 3, \dots, n\} \} \right) \cup \{ \{n - k + 2, n - k + 3, \dots, n\} \}$$

has a smaller weight. Now the a_i 's are determined as in the proof of Theorem 7, with the exception that a_1 must be 0 by Proposition 4. This proves the claim for $\lambda < n - k + 1$.

If $\lambda > n - k + 1$, then $\binom{[n]}{k}$ is the unique MSFA of minimum weight. To see this, assume that $\mathcal{B} \neq \emptyset$, and use $|\nabla \mathcal{B}| \leq (n - k + 1)|\mathcal{B}|$ which implies that

$$\binom{[n]}{k} = (\mathcal{F} \setminus \mathcal{B}) \cup \nabla \mathcal{B}$$

has a smaller weight than \mathcal{F} .

Finally, if $\lambda = n - k + 1$, then choosing $a_1 = 0$ and the other a_i 's as in Theorem 7 (i.e., $a_i = n - k - 1 + i$ for $i = 2, \dots, k$) gives an MSFA that has the same weight as $\binom{[n]}{k}$. \blacksquare

Example 9. We are looking for all FSFA and all MSFA with minimum BLYM value in $\binom{[8]}{6} \cup \binom{[8]}{5}$. That is, $n = 8$ and $k = 6$. By $\alpha = 1/\binom{8}{6} = 1/28$ and $\beta = 1/\binom{8}{5} = 1/56$, we obtain $\lambda = \beta/\alpha = 1/2$.

As $1 + (n - k)/\lambda = 5 < 6$, Theorem 7 yields $a_6 = n - k - 1 + 6 = 7$.

By $1 + 2/\lambda = 5$, we obtain that $a_5 = \lceil 4 \cdot \frac{3}{2} - 1 \rceil$ or $a_5 = \lfloor 4 \cdot \frac{3}{2} \rfloor$, i.e., that $a_5 \in \{5, 6\}$.

Finally, $1/\lambda = 2$ implies $a_4 = 4$, $a_3 = 3$, $a_2 \in \{0, 2\}$, and $a_1 = 0$.

As we have two choices for a_5 and a_2 , respectively, there are four optimal FSFA. By $a_1 = 0$, all of them are also MSFA.

1. With $a_5 = 6$ and $a_2 = 2$, the number of 6-sets is

$$|\mathcal{A}| = \binom{7}{6} + \binom{6}{5} + \binom{4}{4} + \binom{3}{3} + \binom{2}{2} = 16,$$

while the number of 5-sets is

$$|\mathcal{B}| = \binom{8}{5} - [\binom{7}{5} + \binom{6}{4} + \binom{4}{3} + \binom{3}{2} + \binom{2}{1}] = 56 - 45 = 11.$$

The corresponding FSFA is the union of \mathcal{A} , the collection of the first sixteen 6-sets in squashed order, and $\mathcal{B} = \binom{[8]}{5} \setminus \Delta\mathcal{A}$. Its BLYM value is

$$\text{BLYM}(\mathcal{A} \cup \mathcal{B}) = \frac{16}{28} + \frac{11}{56} = \frac{43}{56}.$$

2. With $a_5 = 6$ and $a_2 = 0$ we obtain

$$|\mathcal{A}| = \binom{7}{6} + \binom{6}{5} + \binom{4}{4} + \binom{3}{3} = 15,$$

$$|\mathcal{B}| = \binom{8}{5} - [\binom{7}{5} + \binom{6}{4} + \binom{4}{3} + \binom{3}{2}] = 56 - 43 = 13,$$

$$\text{BLYM}(\mathcal{A} \cup \mathcal{B}) = \frac{15}{28} + \frac{13}{56} = \frac{43}{56}.$$

3. With $a_5 = 5$ and $a_2 = 2$ we obtain

$$|\mathcal{A}| = \binom{7}{6} + \binom{5}{5} + \binom{4}{4} + \binom{3}{3} + \binom{2}{2} = 11,$$

$$|\mathcal{B}| = \binom{8}{5} - [\binom{7}{5} + \binom{5}{4} + \binom{4}{3} + \binom{3}{2} + \binom{2}{1}] = 56 - 35 = 21,$$

$$\text{BLYM}(\mathcal{A} \cup \mathcal{B}) = \frac{11}{28} + \frac{21}{56} = \frac{43}{56}.$$

4. With $a_5 = 5$ and $a_2 = 0$ we obtain

$$|\mathcal{A}| = \binom{7}{6} + \binom{5}{5} + \binom{4}{4} + \binom{3}{3} = 10,$$

$$|\mathcal{B}| = \binom{8}{5} - [\binom{7}{5} + \binom{5}{4} + \binom{4}{3} + \binom{3}{2}] = 56 - 35 = 23,$$

$$\text{BLYM}(\mathcal{A} \cup \mathcal{B}) = \frac{10}{28} + \frac{23}{56} = \frac{43}{56}.$$

■

3 Cases of special interest

Theorem 7 and Corollary 8 together with (2) and $\mathcal{B} = \binom{[n]}{k-1} \setminus \Delta\mathcal{A}$ give the formula

$$w(\mathcal{F}) = \beta\left(\binom{n}{k-1}\right) + \sum_{i=1}^k \left(\alpha\left(\binom{a_i}{i}\right) - \beta\left(\binom{a_i}{i-1}\right) \right) \quad (6)$$

for the smallest weight of an FSFA and an MSFA in $\binom{[n]}{k} \cup \binom{[n]}{k-1}$, where the a_i 's are chosen as in the theorem and the corollary, respectively. (Note that for our formula to be accurate we have to adopt the somewhat unusual convention that $\binom{0}{0}$ is 0.)

3.1 FSFA and MSFA of minimum size

Let $s(n, k)$ denote the minimum size of an FSFA in $\binom{[n]}{k} \cup \binom{[n]}{k-1}$. By Theorem 7 with $\alpha = \beta = 1$, $s(n, k)$ is equal to the right-hand side of (6) for

$$a_i = \begin{cases} n - k - 1 + i & \text{if } i \geq n - k + 2, \\ 2i - 3 \text{ or } 2i - 2 & \text{if } n - k + 1 \geq i \geq 3, \\ 2 & \text{if } i = 2, \\ 0 \text{ or } 1 & \text{if } i = 1. \end{cases}$$

For the minimum size of an MSFA we have to choose $a_1 = 0$ and the other a_i 's as above by Corollary 8. Consequently, $s(n, k)$ is also the minimum size of an MSFA in $\binom{[n]}{k} \cup \binom{[n]}{k-1}$.

Using the above values for the a_i 's in (6) gives the following formula for $s(n, k)$.

Corollary 10. *Let $1 \leq k \leq (n+1)/2$. Then*

$$s(n, k) = s(n, n - k + 1) = \binom{n}{k-1} - \sum_{i=1}^{k-1} \frac{1}{i+1} \binom{2i}{i}.$$

Corollary 10 implies that as n gets larger for fixed k the optimal FSFA look more and more like the $(k-1)$ -st level of B_n . According to the next corollary, this remains true if we allow k to depend on n .

Corollary 11. *For any $k = k(n) \leq (n+1)/2$ one has $s(n, k) = (1 + o(1)) \binom{n}{k-1}$.*

Proof: We need to show that

$$\sum_{i=1}^{k-1} \frac{1}{i+1} \binom{2i}{i} = o\left(\binom{n}{k-1}\right).$$

Let

$$p_i = \frac{1}{i+1} \binom{2i}{i}$$

be the i -th term in this sum. Then

$$\frac{p_i}{p_{i-1}} = \frac{4(i - \frac{1}{2})}{i+1} = 4 \left(1 - \frac{3/2}{i+1}\right),$$

so that, for $\ell \in \{1, \dots, k-2\}$,

$$\frac{p_{k-1}}{p_{k-1-\ell}} = 4^\ell \prod_{j=0}^{\ell-1} \left(1 - \frac{3/2}{k-j}\right) \geq 2^\ell$$

where the inequality follows from the fact that every factor in the product is at least $1/2$. Therefore,

$$\sum_{i=1}^{k-1} p_i \leq p_{k-1} \sum_{\ell=0}^{k-1} (1/2)^\ell < 2p_{k-1}.$$

Furthermore, using $2k \leq n+1$,

$$\begin{aligned} \frac{p_{k-1}}{\binom{n}{k-1}} &= \frac{1}{k} \cdot \frac{(2k-2)!}{((k-1)!)^2} \cdot \frac{(k-1)!(n-k+1)!}{n!} = \frac{(2k-2)!(n-k+1)!}{k!n!} \\ &= \frac{1}{n} \cdot \frac{k+1}{n-k+2} \cdot \frac{k+2}{n-k+3} \cdots \frac{2k-2}{n-1} \leq \frac{1}{n}, \end{aligned}$$

and consequently,

$$\sum_{i=1}^{k-1} \frac{1}{i+1} \binom{2i}{i} < 2p_{k-1} \leq \frac{2}{n} \binom{n}{k-1} = o\left(\binom{n}{k-1}\right).$$

■

3.2 FSFA and MSFA of minimum volume

Using $\alpha = k$ and $\beta = k-1$ in Theorem 7 gives a characterization of all minimum volume FSFA in $\binom{[n]}{k} \cup \binom{[n]}{k-1}$. As $\lambda = (k-1)/k < 1$ in this case, these FSFA are all also MSFA.

3.3 FSFA and MSFA of minimum BLYM value

To find the minimum BLYM value of an FSFA or MSFA in $\binom{[n]}{k} \cup \binom{[n]}{k-1}$, we can use (6) with $\alpha = 1/\binom{n}{k}$ and $\beta = 1/\binom{n}{k-1}$ which means that $\lambda = (n-k+1)/k$. Note that the optimal FSFA given by Theorem 7 are also MSFA if $k > (n+1)/2$. For $k = (n+1)/2$ there is an optimal FSFA which also is an MSFA, but not for $k \leq n/2$.

Let $\text{BLYM}(n, k)$ be the minimum BLYM value of an MSFA in $\binom{[n]}{k} \cup \binom{[n]}{k-1}$. As the optimal FSFA and MSFA differ only marginally, it is easy to verify that the following asymptotic result still holds for FSFA. For brevity, we only look at MSFA here.

Corollary 12. *For fixed $k \geq 1$ one has $\lim_{n \rightarrow \infty} \text{BLYM}(n, k) = 1 - \frac{(k-1)^{k-1}}{k^k}$.*

Proof: For the asymptotic to be shown, we can assume that $\lambda = (n-k+1)/k$ is large. Considering this, Corollary 8 implies that for an optimal MSFA we can choose $a_1 = 0$ and $a_i = \lfloor (i-1)(\lambda+1) \rfloor$ for $i = 2, \dots, k$.

For $2 \leq i \leq k-1$ we have $a_i = \lfloor (i-1)(\lambda+1) \rfloor = \lfloor \frac{i-1}{k}(n+1) \rfloor \leq \frac{i}{k}n < n$. Consequently, for $i \neq k$ the summands

$$\alpha \binom{a_i}{i} - \beta \binom{a_i}{i-1} = \frac{\binom{a_i}{i}}{\binom{n}{k}} - \frac{\binom{a_i}{i-1}}{\binom{n}{k-1}}$$

on the right-hand side of (6) all tend to 0 as $n \rightarrow \infty$.

The claim follows by $\beta \binom{n}{k-1} = 1$ and the by fact that

$$\begin{aligned} \alpha \binom{a_k}{k} - \beta \binom{a_k}{k-1} &= \frac{\binom{\lfloor \frac{k-1}{k}(n+1) \rfloor}{k}}{\binom{n}{k}} - \frac{\binom{\lfloor \frac{k-1}{k}(n+1) \rfloor}{k-1}}{\binom{n}{k-1}} \\ &= \left(\frac{\lfloor \frac{k-1}{k}(n+1) \rfloor - k + 1}{n - k + 1} - 1 \right) \prod_{j=0}^{k-2} \frac{\lfloor \frac{k-1}{k}(n+1) \rfloor - j}{n - j} \end{aligned}$$

tends to

$$\left(\frac{k-1}{k} - 1 \right) \left(\frac{k-1}{k} \right)^{k-1} = - \frac{(k-1)^{k-1}}{k^k}$$

as $n \rightarrow \infty$. ■

4 Open Problems

There are several open problems which have a significant relationship to the contents of this paper. Some of these are new problems, and these are stated below.

Problem 1. Characterise those MSFA that simultaneously attain more than one of minimum size, minimum volume, and minimum BLYM value.

Problem 2. Is there an MSFA which has an equivalent non-flat antichain (w.r.t. the equivalence relation in the introduction)? We conjecture the answer to be negative.

Problem 3. A generalised maximal squashed flat antichain (GMSFA) is a maximal squashed antichain which contains sets of exactly two sizes which do not need to be consecutive. Determine the minimum size, volume, and BLYM values and answer the above questions in this more general setting.

Problem 4. Consider the analogous questions for FSFA.

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